Problem 2.36

Consider the following quote from Galileo's *Dialogues Concerning Two New Sciences:*

> Aristotle says that "an iron ball of 100 pounds falling from a height of one hundred cubits reaches the ground before a one-pound ball has fallen a single cubit." I say that they arrive at the same time. You find, on making the experiment, that the larger outstrips the smaller by two finger-breadths, that is, when the larger has reached the ground, the other is short of it by two finger-breadths.

We know that the statement attributed to Aristotle is totally wrong, but just how close is Galileo's claim that the difference is just "two finger breadths"? **(a)** Given that the density of iron is about 8 g/cm^3, find the terminal speeds of the two iron balls. **(b)** Given that a cubit is about 2 feet, use Equation (2.58) to find the time for the heavier ball to land and then the position of the lighter ball at that time. How far apart are they?

Solution

Part (a)

A large and fast-moving object, such as a falling iron ball, is subject to quadratic air resistance, and the terminal speed for it is

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}.$$

For spherical objects, $c = \gamma D^2$. In particular, in air at STP, $\gamma = 0.25\ \text{N}\cdot\text{s}^2/\text{m}^4$.

$$v_{\text{ter}} = \sqrt{\frac{mg}{0.25D^2}} = \frac{2}{D}\sqrt{mg} \tag{1}$$

Since the diameter isn't known, use the density to write it in terms of the mass.

$$\begin{aligned} m &= \varrho V \\ &= \varrho\left[\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right] \end{aligned}$$

Solve for $D/2$.

$$\sqrt[3]{\frac{3m}{4\pi\varrho}} = \frac{D}{2}$$

As a result, equation (1) becomes

$$v_{\text{ter}} = \sqrt[3]{\frac{4\pi\varrho}{3m}}\sqrt{mg} = \sqrt[3]{\frac{4\pi\varrho}{3}}\sqrt[6]{m}\sqrt{g}.$$

The terminal velocity of the hundred-pound iron ball is

$$v_{\text{ter100}} = \sqrt[3]{\frac{4\pi}{3}\left[8\ \frac{\text{g}}{\text{cm}^3} \times \frac{1\ \text{kg}}{1000\ \text{g}} \times \left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3\right]} \sqrt[6]{100\ \text{lb} \times \frac{1\ \text{kg}}{2.204622476\ \text{lb}}} \sqrt{9.81\ \frac{\text{m}}{\text{s}^2}} \approx 191\ \frac{\text{m}}{\text{s}},$$

and the terminal velocity of the one-pound iron ball is

$$v_{\text{ter1}} = \sqrt[3]{\frac{4\pi}{3}\left[8\ \frac{\text{g}}{\text{cm}^3} \times \frac{1\ \text{kg}}{1000\ \text{g}} \times \left(\frac{100\ \text{cm}}{1\ \text{m}}\right)^3\right]} \sqrt[6]{1\ \text{lb} \times \frac{1\ \text{kg}}{2.204622476\ \text{lb}}} \sqrt{9.81\ \frac{\text{m}}{\text{s}^2}} \approx 88.5\ \frac{\text{m}}{\text{s}}.$$

Part (b)

Equation (2.58) is on page 61 and gives the position for a projectile falling down (released from rest) in a medium with quadratic air resistance.

$$y(t) = \frac{v_{\text{ter}}^2}{g} \ln\left[\cosh\left(\frac{g}{v_{\text{ter}}} t\right)\right] \tag{2.58}$$

Solve it for $t = t_0$ to find how long it takes the heavier ball to fall from a height h of 100 cubits.

$$h = \frac{v_{\text{ter}}^2}{g} \ln\left[\cosh\left(\frac{g}{v_{\text{ter}}} t_0\right)\right]$$

$$\frac{gh}{v_{\text{ter}}^2} = \ln\left[\cosh\left(\frac{g}{v_{\text{ter}}} t_0\right)\right]$$

$$\exp\left(\frac{gh}{v_{\text{ter}}^2}\right) = \cosh\left(\frac{g}{v_{\text{ter}}} t_0\right)$$

$$\cosh^{-1}\left[\exp\left(\frac{gh}{v_{\text{ter}}^2}\right)\right] = \frac{g}{v_{\text{ter}}} t_0$$

$$t_0 = \frac{v_{\text{ter}}}{g} \cosh^{-1}\left[\exp\left(\frac{gh}{v_{\text{ter}}^2}\right)\right]$$

Therefore, the time it takes for the hundred-pound iron ball to reach the ground is

$$t_0 \approx \frac{191\ \frac{\text{m}}{\text{s}}}{9.81\ \frac{\text{m}}{\text{s}^2}} \cosh^{-1}\left\{\exp\left[\frac{\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)\left(100\ \text{cubits} \times \frac{2\ \text{ft}}{1\ \text{cubit}} \times \frac{1\ \text{m}}{3.281\ \text{ft}}\right)}{\left(191\ \frac{\text{m}}{\text{s}}\right)^2}\right]\right\} \approx 3.53\ \text{seconds}.$$

Plug this time into Equation (2.58) to determine how far the one-pound ball has fallen at impact.

$$y(t_0) \approx \frac{\left(88.5\ \frac{\text{m}}{\text{s}}\right)^2}{9.81\ \frac{\text{m}}{\text{s}^2}} \ln\left\{\cosh\left[\frac{9.81\ \frac{\text{m}}{\text{s}^2}}{88.5\ \frac{\text{m}}{\text{s}}}(3.53\ \text{seconds})\right]\right\} \approx 59.8\ \text{m}$$

Therefore, the one-pound iron ball is

$$h - y(t_0) \approx 100\ \text{cubits} \times \frac{2\ \text{ft}}{1\ \text{cubit}} \times \frac{1\ \text{m}}{3.281\ \text{ft}} - 59.8\ \text{m} \approx 1.17\ \text{m},$$

more than a meter, off the ground when the hundred-pound iron ball strikes it.